Nonequilibrium effect of the turbulent-energy-production process on the inertial-range energy spectrum

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The importance of the nonequilibrium effect of the turbulent-energy-production process is discussed in the context of the inertial-range energy spectrum. A dynamical property of the process related to the Lagrange derivative of the turbulent-energy transfer rate gives rise to an important deviation of the inertial-range spectrum from Kolmogorov's $-\frac{5}{3}$ power law at the energy-containing side. Through this deviation, the turbulent-viscosity concept possesses a nonequilibrium character that is indispensable for describing statistically highly time- and space-dependent properties of turbulence. It is concluded that the inertial-range energy spectrum cannot be specified using only the energy transfer rate.

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I. INTRODUCTION

Theoretical studies of turbulence have been done mainly using two approaches. One approach aims at clarifying the similarity properties of small-scale structures of turbulence. The representative finding is Kolmogorov's scaling law [1] leading to the $-\frac{5}{3}$ power law for the inertial-range spectrum. The derivation of this power law using the Navier-Stokes equation has been one of the main purposes of two-point closure theories such as Kraichnan's direct-interaction approximation (DIA) [2,3].

Experimental studies of small-scale turbulence structures have revealed the breakdown of the Kolmogorov scaling law, specifically, in the velocity-derivative statistics to which fine-scale fluctuations mainly contribute. In order to avoid this difficulty, various models based on different probabilistic properties of the local energy dissipation rate have been proposed. The representative examples are the log-normal model of Kolmogorov [4] and the β model of Frisch, Sulem, and Nelkin [5]. The validity of these models was discussed by Nakano and Nelkin [6] using a dynamical scaling argument. More elaborate models based on the multifractal concept have recently been developed to incorporate the detailed statistical properties of the local energy dissipation rate (see [7] and the references cited therein).

Another approach to turbulence is the one-point turbulence modeling [8] for the study of turbulent shear flows that are important in engineering and scientific flow phenomena. Such modeling is roughly classified into the turbulent-viscosity-type and second-order modelings. In the former, the Reynolds stress in the mean Navier-Stokes equation is modeled with the aid of the turbulent-viscosity concept. In the latter, the Reynolds stress is directly dealt with and the higher-order correlation functions in its transport equation are modeled. In these modelings, dimensional and tensor analysis and the invariance properties of the Navier-Stokes equation have been a guiding principle, and little use has been made of the detailed turbulence statistics in the wave-number space.

There has not always been much contact between the preceding two approaches. During these ten years, many efforts have been made towards the construction of the one-point turbulence models using two-point closure theories that were originally developed for the study of isotropic turbulence. The examples of such work are a two-scale DIA (TSDIA) [9,10] and the renormalization-group (RNG) method [11,12]. In these studies, the models of the turbulent-viscosity type are first derived [9,11,12] and then the second-order models are obtained through the renormalization of the former [10]. Kolmogorov's $-\frac{5}{3}$ power law for the energy spectrum plays a central role in deriving the turbulent-viscosity-type models, specifically, in the estimate of model constants.

In the context of the one-point modeling, the $-\frac{5}{3}$ power law is closely connected with the equilibrium turbulent-viscosity approximation to the Reynolds stress. The prominent feature of the power law lies in the equilibrium hypothesis of the inertial range; namely, the range is supplied with energy by the energy-containing range with ε as the energy transfer or injection rate, whereas this transferred energy is changed into heat with the same rate in the dissipation range. However, strong doubt has recently been raised about the validity of he equilibrium form of the turbulent-viscosity expression in such a nonequilibrium turbulence state as is encountered in homogeneous shear turbulence. This fact also casts doubt on the equilibrium hypothesis of the inertial range in a sense different from the effect of intermittency.

In this work, we shall pay special attention to the nonequilibrium effect of the turbulent-energy-production process on the inertial range. We shall discuss the energy spectrum in turbulent shear flows with the aid of the results of the TSDIA [9]. We shall make the discussion from the viewpoint of the effect of the inertial-range spectrum on the Reynolds stress, specifically, on the turbulent viscosity. This approach is rather indirect, compared with the comparison of the energy spectrum with the direct numerical simulation (DNS) results of a turbulent

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shear flow. The importance of the nonequilibrium effect, however, will be understood more clearly in the sense that the effect on the inertial-range spectrum manifests itself in the one-point turbulence models widely used in the study of real-world turbulent flows. On the basis of this discussion, we shall show that the energy-containing side of the inertial range is strongly affected by the dynamical property of the turbulent-energy-production process that is related to the Lagrange derivative of the energy transfer rate.

The present paper is organized as follows. The fundamental equations are given in Sec. II and the inertial-range energy spectrum in isotropic turbulence is outlined in Sec. III. The relationship of Kolmogorov's $\frac{5}{3}$ power law with the isotropic turbulent viscosity in the one-point turbulence modeling is discussed in Sec. IV. The energy spectrum of the inertial range that includes the effect of nonequilibrium properties is given in Sec. V. The implications of this spectrum are discussed in light of homogeneous shear turbulence in Sec. VI. The concluding remarks are given in Sec. VII.

II. FUNDAMENTAL EQUATIONS

The fluid motion at low Mach numbers obeys the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+\nu\Delta\mathbf{u} , \tag{1}$$

with the solenoidal condition $\nabla\cdot\mathbf{u}=0$, where $\mathbf{u}$ is the velocity, p is the pressure divided by constant density, and ν is the kinematic viscosity.

We introduce the ensemble mean $\langle\ \rangle$ to divide a quantity f into the mean F $(=\langle f\rangle)$ and the fluctuation f' $(=f-F)$, where $f=(\mathbf{u},p)$, $F=(\mathbf{U},P)$, and $f'=(\mathbf{u}',p')$. Then the mean Navier-Stokes equation is given by

$$\frac{D\mathbf{U}}{Dt}=-\nabla P+\nabla\cdot R+\nu\Delta\mathbf{U} , \tag{2}$$

with $\nabla\cdot\mathbf{U}=0$ and $D/Dt=\partial/\partial t+\mathbf{U}\cdot\nabla$. Here R is the Reynolds-stress tensor defined as

$$R_{ij}=-\langle u_i'u_j'\rangle \tag{3}$$

$[(\nabla\cdot R)_i=(\partial/\partial x_j)R_{ji}]$.

In the one-point turbulence modeling, R_{ij} is often modeled as

$$R_{ij}=-\tfrac{2}{3}K\delta_{ij}+\nu_T S_{ij} , \tag{4}$$

with the aid of the turbulent-viscosity concept. Here $K(=\langle\mathbf{u}'^2/2\rangle)$ is the turbulent energy and $S_{ij}(=\partial U_j/\partial x_i+\partial U_i/\partial x_j)$ is the mean-velocity-strain tensor. The turbulent viscosity ν_T is most typically written as

$$\nu_T=C_\nu\frac{K^2}{\varepsilon},\quad C_\nu=0.09 , \tag{5}$$

using K and its dissipation rate $\varepsilon=\langle\nu(\partial u_j'/\partial x_i)^2\rangle$.

The transport equations for K and the mean-velocity counterpart $K_M=(\mathbf{U}^2/2)$ are given by

$$\frac{DK}{Dt}=P_K-\varepsilon+\nabla\cdot\mathbf{T}_K+\nu\Delta K , \tag{6}$$

$$\frac{DK_M}{Dt}=-\left[P_K+\nu\left[\frac{\partial U_j}{\partial x_i}\right]^2\right]+\nabla\cdot(-P\mathbf{U}+R\mathbf{U})+\nu\Delta K_M, \tag{7}$$

where

$$P_K=R_{ij}\frac{\partial U_j}{\partial x_i} , \tag{8}$$

$$\mathbf{T}_K=-\left\langle\left[\frac{\mathbf{u}'^2}{2}+p'\right]\mathbf{u}'\right\rangle . \tag{9}$$

Here P_K is usually called the turbulent-energy-production rate. Since ε is non-negative, K cannot be sustained under vanishing mean-velocity gradients so long as the turbulent energy is not supplied through boundaries. On the other hand, P_K also plays a role of energy sink in the K_M equation (7). The mean motion is supplied with energy by the P- (mean pressure) related term in Eq. (7) that represents the energy injection through the imposition of external forces on boundaries. To put it briefly, the energy supplied through the pressure P cascades towards u' by the turbulence effect or P_K and is finally converted to heat with the rate.

III. INERTIAL-RANGE ENERGY SPECTRUM FOR ISOTROPIC TURBULENCE

In homogeneous isotropic turbulence with vanishing $\nabla\mathbf{U}$, attention is focused on $\mathbf{u}'$. In this situation, P_K vanishes and the turbulence state decays in time or is sustained through the energy injection due to a random force, etc. The fluctuation $\mathbf{u}'$ is expressed in the Fourier integral as

$$\mathbf{u}'=\int\mathbf{u}'(\mathbf{k};t)\exp(-i\mathbf{k}\cdot\mathbf{x})d\mathbf{k} . \tag{10}$$

Statistical properties of turbulence are characterized using the correlation functions of $\mathbf{u}'(\mathbf{k};t)$.

The most typical second-order correlation function of $\mathbf{u}'(\mathbf{k};t)$ is the energy spectrum, which is defined as

$$K=\int E(k)dk . \tag{11}$$

From Eq. (10), $E(k)$ is given by

$$E(k)\delta(\mathbf{k}+\mathbf{k}')=2\pi k^2\langle u_i'(\mathbf{k};t)u_j'(\mathbf{k}';t)\rangle \tag{12}$$

[$\delta(\mathbf{k})$ is the Dirac delta function].

A. Kolmogorov's scaling law

The range of $E(k)$ in isotropic turbulence at high Reynolds numbers is roughly divided into three ranges: the energy-containing, inertial, and dissipation ranges (see Fig. 1, in which l_C is the length scale characterizing the energy-containing range, and $l_D[=(\nu^3/\varepsilon)^{1/4}]$ is the dissipation length scale). In the case of turbulent shear flows with nonvanishing $\nabla\mathbf{U}$, the energy-containing range of $\mathbf{u}'$ is associated with the K production mechanism that is

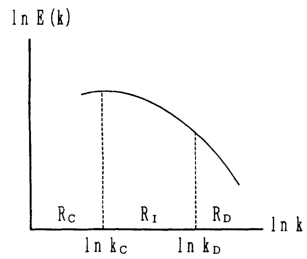

FIG. 1. Schematic classification of the energy-spectrum range: R_C, energy-containing range; R_I, inertial range; R_D, dissipation range ($k_C = l_C^{-1}$ and $k_D = l_D^{-1}$).

given by P_K [Eq. (8)]. Within the framework of isotropic turbulence without such a mechanism, this range is merely described through the small-wave-number behaviors of $E(k)$. Therefore the existence of the universal properties that are independent of the detailed properties of $E(k)$ at small wave numbers is a primary goal of the study of isotropic turbulence.

In Kolmogorov's scaling law [1], the energy transfer rate from the energy-containing to inertial ranges is the sole quantity characterizing the effect of the former. In the inertial range, no energy dissipation occurs and therefore the rate is essentially equal to the dissipation rate. Under this concept, $E(k)$ is written as

$$E(k) = K_0 \varepsilon^{2/3} k^{-5/3} , \tag{13}$$

where K_0 is the so-called Kolmogorov constant. One of the major purposes of two-point closure theories has long been the analytical derivation of Eq. (13).

B. Intermittency effects

At the stage of the preceding section, it is not clear which of the following two processes is more important for the inertial range: (a) the energy dissipation process resulting in the energy transfer from the inertial to dissipation ranges and (b) the energy injection process from the energy-containing to inertial ranges. If the process (a) is more important, the fluctuation effect of the local energy dissipation rate ε' [$=\nu(\partial u_j'/\partial x_i)^2$] around ε, which is called the intermittency effect, may become an important factor generating the deviation of $E(k)$ and the velocity-derivative statistics from Kolmogorov's scaling law. This standpoint leads to the log-normal model [4], the β model [5], etc.

The effect of intermittency on $E(k)$ is written as

$$E(k) = K_0 \varepsilon^{2/3} k^{-5/3} (k l_C)^{-\mu} . \tag{14}$$

Here l_C is the length scale characterizing the energy-containing range, but it cannot be defined uniquely within the framework of the current intermittency arguments. We may adopt the correlation length of $\mathbf{u}'$ as a typical example of l_C. The magnitude of the parameter μ is dependent on the statistical property of ε' and is estimated to be about 0.17 from the observations. As a result, the intermittency effect on $E(k)$ is not so important and becomes small, specifically at the energy-containing side of the inertial range or near $k = l_C^{-1}$. This is the reason ε has long been considered to be almost the sole quantity determining the effect of the energy-containing range on the inertial-range energy spectrum.

In the later discussion, we shall take the second standpoint (b); namely, the turbulent-energy-production process, specifically, its dynamical or nonequilibrium property, is very important for $E(k)$ of the inertial range.

IV. RELATIONSHIP OF THE KOLMOGOROV SPECTRUM WITH ISOTROPIC TURBULENT VISCOSITY

In Sec. II, we noted that Eq. (5) is widely used as the turbulent viscosity in the one-point expression for the Reynolds stress (4). Let us consider Eq. (5) in light of the Kolmogorov energy spectrum (13). Since $K^{1/2}$ and l_C are the velocity and the length scales characterizing the energy-containing eddies, we may write the turbulent viscosity ν_T as

$$\nu_T \cong K^{1/2} l_C , \tag{15}$$

apart from the numerical factor. From the fact that Eq. (13) with $k \cong l_C^{-1}$ corresponds to the energy-containing-range energy spectrum at the inertial-range side, K is estimated as

$$K \cong l_C^{-1} E(l_C^{-1}) \cong \varepsilon^{2/3} l_C^{2/3} . \tag{16}$$

From Eq. (16), we have

$$l_C \cong K^{3/2}/\varepsilon . \tag{17}$$

The combination of Eq. (15) with Eq. (17) leads us to the widely used turbulent viscosity (5).

The above discussion shows that Eq. (5) is founded on the Kolmogorov hypothesis of the equilibrium energy transfer from the energy-containing to dissipation ranges through the inertial one. This point is also clear from the derivation of Eq. (4) with Eq. (5) using the TSDIA [9] and the RNG [11,12], in which the Kolmogorov spectrum (13) plays a central role. In these methods, the isotropic velocity fluctuation interacts with the mean-velocity gradient to generate an anisotropic one. The latter leads to the second term of R_{ij} [Eq. (4)] with Eq. (5).

The comparison with the experimental and DNS results has already revealed some critical deficiencies of Eq. (4) with Eq. (5). Such a typical instance is that it cannot cope with the strong nonequilibrium state of turbulence as is encountered in a flow under strong adverse pressure gradients and homogeneous shear turbulence. In the case of homogeneous shear turbulence discussed later, the turbulence state represented by K and ε is spatially uniform, but it continues to develop in time. As a result, the process of energy transfer from the mean motion to the inertial range through the energy-containing one is not in an equilibrium state; namely, the energy-containing range is greatly affected by $\partial\varepsilon/\partial t$ in addition to ε that plays a central role in Kolmogorov's scaling law. In reality, the difficulty with the equilibrium turbulent viscosity (5) in homogeneous shear turbulence can be overcome through the inclusion of the nonequilibrium effect like $\partial\varepsilon/\partial t$ [13], as will be referred to later.

In what follows, we shall discuss the relationship of the inertial-range energy spectrum with the nonequilibrium state of the energy-containing range.

V. INERTIAL-RANGE ENERGY SPECTRUM UNDER NONEQUILIBRIUM EFFECTS

In the stationary state of isotropic turbulence, ε is constant in space and time. However, we can allow ε in the Kolmogorov spectrum (13) to depend on space and time as long as its change is gradual. The condition for the slow spatial change of ε may be written as

$$k^{-1} \ll l_{SI} \quad \text{for } l_D < k^{-1} < l_C . \tag{18}$$

Here l_{SI} is defined as

$$l_{SI} = \varepsilon/|\nabla\varepsilon| \tag{19}$$

and represents a length scale characterizing the spatial inhomogeneity of ε. Equation (5), based on the Kolmogorov spectrum, is applicable to the weakly nonequilibrium state of turbulence satisfying Eq. (18).

The turbulent-viscosity approximation (4) with Eq. (5) has been applied with some success to the analyses of a pipe flow, a wall flow, etc. These types of flows are in an equilibrium state at least in the mainstream direction; namely, the turbulence quantities, such as K and ε, do not change in the direction. On the other hand, the performance of the approximation is poor in a turbulent flow under strong adverse pressure gradients, a flow along a curved boundary, etc. in addition to highly nonstationary turbulent flows like homogeneous shear turbulence. In the former flows, the change of turbulence quantities in the mainstream direction is very large. This fact suggests that in place of the Kolmogorov spectrum (13), the energy spectrum with nonequilibrium properties of turbulence included is necessary for their description.

A. TSDIA formalism

The author [9] previously proposed a two-point closure method for the study of turbulent shear flows, which is the combination of the DIA with a multiple-scale method. In the TSDIA, we introduce two space and time scales, that is, the fast $(\boldsymbol{\xi},\tau)$ and slow $(\mathbf{X},T)$ variables:

$$\boldsymbol{\xi}(=\mathbf{x}), \quad \tau(=t); \quad \mathbf{X}(=\delta\mathbf{x}), \quad T(=\delta t) , \tag{20}$$

using a scale parameter δ that is assumed to be small. Then a quantity f is written as

$$f = F(\mathbf{X};T) + f'(\boldsymbol{\xi},\mathbf{X};\tau,T) , \tag{21}$$

and f' is expressed in the Fourier representation of the fast space variable $\boldsymbol{\xi}$:

$$f'(\boldsymbol{\xi},\mathbf{X};\tau,T) = \int f'(\mathbf{k},\mathbf{X};\tau,T)\exp(-i\mathbf{k}\cdot\boldsymbol{\xi})d\mathbf{k} . \tag{22}$$

Moreover we expand f' in δ as

$$f'(\mathbf{k},\mathbf{X};\tau,T) = \sum_{n=0} \delta^n f'_n(\mathbf{k},\mathbf{X};\tau,T) . \tag{23}$$

In the TSDIA, the equation for $\mathbf{u}'_0$ does not depend directly on the mean-velocity-gradient tensor $\nabla_X \mathbf{U}[\nabla_X = (\partial/\partial X_i; i=1,2,3]$. The $\nabla_X\mathbf{U}$-related effects as well as the Lagrange derivative $D\mathbf{u}'_0/DT$ appear through $\mathbf{u}'_n$ $(n \geq 1)$ $(D/DT = \partial/\partial T + \mathbf{U}\cdot\nabla_X)$. As a result, the Kolmogorov spectrum (13) may be extended to an expression with the weak dependence of ε on space and time through the slow variables $\mathbf{X}$ and T.

B. Energy spectrum with the nonequilibrium property included

The primary part of the energy spectrum given by the TSDIA [9] is divided into the equilibrium part leading to the Kolmogorov spectrum E_K and the nonequilibrium one E_N:

$$E = E_K + E_N , \tag{24}$$

where

$$E_K = 4\pi k^2 Q(k,\mathbf{x};\tau,\tau,t) , \tag{25}$$

$$E_N = -4\pi k^2 \int_{-\infty}^{\tau} G(k,\mathbf{x};\tau,\tau_1,t)\frac{D}{Dt}Q(k,\mathbf{x};\tau,\tau_1,t)d\tau_1 . \tag{26}$$

Here we should note that the replacement of $\mathbf{X}$ and T with $\delta\mathbf{x}$ and δt in the final stage of the analysis leads to the automatic disappearance of the parameter δ. In Eqs. (25) and (26), Q is the covariance of $\mathbf{u}'_0$ defined by

$$Q(k,\mathbf{x};\tau,\tau',t)\delta(\mathbf{k}+\mathbf{k}') = \tfrac{1}{2}\langle u'_{0i}(\mathbf{k},\mathbf{x};\tau,t)u'_{0i}(\mathbf{k}',\mathbf{x};\tau',t)\rangle , \tag{27}$$

and G is the response function for $\mathbf{u}'_0$ (see [9] for the details).

In order to see the relationship with the Kolmogorov spectrum (13), we introduce the simple expressions for Q and G:

$$Q(k,\mathbf{x};\tau,\tau',t) = C_S\varepsilon^{2/3}k^{-11/3} \times\exp(-C_W\varepsilon^{1/3}k^{2/3}|\tau-\tau'|) , \tag{28}$$

$$G(k,\mathbf{x};\tau,\tau',t) = H(\tau-\tau')\exp[-C_W\varepsilon^{1/3}k^{2/3}(\tau-\tau')] , \tag{29}$$

with $\varepsilon = \varepsilon(\mathbf{x};t)$, where C_S and C_W are numerical constants, and $H(\tau-\tau')$ is the step function.

The constants C_S and C_W should be determined from the analysis of the $\mathbf{u}'_0$ equation. On applying the DIA formalism to the equation straightforwardly within the Eulerian framework, we encounter the difficulty of the infrared divergence in the G equation, as is known well in the study of two-point closure methods for isotropic turbulence. In order to avoid this difficulty, the Lagrangian treatment of the $\mathbf{u}'_0$ equation has been proposed since the pioneering work of Kraichnan (see [3] and the references cited therein). In this work, however, we make use of the result that was obtained by a simple method; namely, we remove the infrared divergence in the G equation so that the resulting Kolmogorov constant K_0 $(=4\pi C_S)$ should be consistent with the observational counterpart, which scatters around 1.5. As a result, we have [14]

$$C_S=0.12, \quad C_W=0.42 . \tag{30}$$

From Eqs. (24)–(26) and (28)–(30), we have

$$E(k)=K_0\varepsilon^{2/3}k^{-5/3}\left[1-C_N\varepsilon^{-4/3}\frac{D\varepsilon}{Dt}k^{-2/3}\right] , \tag{31}$$

where $K_0=1.5$ and $C_N=0.60$ Equation (31) shows that the deviation of the inertial-range energy spectrum generated by the nonequilibrium effect is given by a $-\frac{7}{3}$ power correction. In the case of homogeneous shear turbulence, the existence of a $-\frac{7}{3}$ power law in the shear-stress spectrum was also discussed by Leslie [15].

Let us introduce the length scale l_P that characterizes a dynamical property of the turbulent-energy-production process:

$$l_P=\varepsilon^2\Big/\left|\frac{D\varepsilon}{Dt}\right|^{3/2} . \tag{32}$$

Using Eq. (32), we can rewrite Eq. (31) as

$$E(k)=K_0\varepsilon^{2/3}k^{-5/3}\left[1-C_N\mathrm{sgn}\left[\frac{D\varepsilon}{Dt}\right](kl_P)^{-2/3}\right] , \tag{33}$$

where $\mathrm{sgn}(A)=1$ and -1 for $A>0$ and $A<0$, respectively. In Sec. VI, we shall show the importance of the l_P-related term in turbulent shear flows. Through the discussion, the physical meaning of l_p will be clarified.

VI. DISCUSSIONS

In Sec. V, it was shown that the nonequilibrium property of the inertial-range spectrum is related to the Lagrange derivative of ε. In engineering and scientific flow phenomena, there are a number of turbulent flows in which the Lagrange derivatives of K, ε, etc. do not vanish. Such flows, however, are geometrically complicated and it is difficult to obtain the detailed properties of turbulence quantities such as ε by using experimental and DNS methods.

A typical instance of the flows whose geometrical structures are simple and which still retain part of the D/Dt effect is homogeneous shear turbulence. In this sense, the turbulence quantities such as K and ε are spatially constant, but they continue to develop in time, and $D\varepsilon/Dt$ is retained as $\partial\varepsilon/\partial t$. Several DNS's of homogeneous shear turbulence have already been done since the work of Rogers and Moin [16] and the temporal behaviors of turbulence quantities have been examined in detail. In what follows, we shall make use of the DNS results of Matsumoto, Nagano, and Tsuzi [17,18] to see the role of the nonequilibrium or D/Dt effect in Eq. (33). Here it is not simple to use the DNS results to clearly identify with the deviation from the Kolmogorov law (13) in the power-law form since the energy-containing side of the inertial range is generally contaminated by various effects associated with the turbulent-energy-production process. So we capture the D/Dt effect in Eq. (33) through the contribution to the turbulent viscosity although such a verification is rather indirect.

A. Breakdown of the equilibrium turbulent viscosity

In homogeneous shear turbulence with the mean velocity $(Sy,0,0)$, the time development of K is described by

$$\frac{\partial K}{\partial t}=P_K-\varepsilon \tag{34}$$

from Eq. (6) (S is the mean shear rate). Under the turbulent-viscosity approximation (4), we have

$$P_K=\nu_T S^2 \tag{35}$$

from Eq. (8).

On using Eq. (34) to examine the validity of Eq. (5) for ν_T, we have two approaches. One is to use the widely adopted model equation in the K-ε model, which is given by

$$\frac{\partial\varepsilon}{\partial t}=C_{\varepsilon 1}\frac{\varepsilon}{K}P_K-C_{\varepsilon 2}\frac{\varepsilon^2}{K} , \tag{36}$$

where $C_{\varepsilon 1}=1.43$ and $C_{\varepsilon 2}=1.9$. Another is the use of the DNS database as ε. Both the approaches lead to a similar conclusion; namely, the time development of K obtained using Eq. (5) as ν_T is entirely different from the DNS counterpart. In Fig. 2, the results based on Eqs. (5) and (34)–(36) are shown in the comparison with the DNS data [13]. Here the initial strain rate SK_0/ε_0 is 28.3 (the initial values K_0 and ε_0 are 0.2). This result indicates that the Kolmogorov spectrum leading to Eq. (5) suffers from a serious drawback at the energy-containing side.

In order to avoid the difficulty with the equilibrium turbulent viscosity (5), the inclusion of the D/Dt effect in ν_T has recently been proposed with the aid of the results of the TSDIA [13]. Namely, Eq. (5) was replaced with a simple nonequilibrium turbulent-viscosity model

$$\frac{\nu_T}{\nu_{TK}}=1-C\frac{K}{\varepsilon}\frac{1}{\nu_{TK}}\frac{D\nu_{TK}}{Dt} , \tag{37}$$

where ν_{TK} corresponds to ν_T of Eq. (5) and $C\cong 1.3$ [the subscript K of ν_{TK} means the correspondence to the Kolmogorov spectrum (13)]. We use this model at $St=2$ to avoid effects of the artificial initial conditions of the

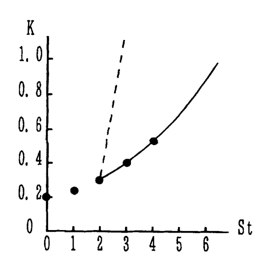

FIG. 2. Time development of K from $St=2$ [13]: ●, DNS result [17,18]; — — —, K-ε model based on Eq. (5); ——, result based on Eq. (37) (S is the mean shear rate and t is the time).

DNS. The result is also plotted in Fig. 2, and the difficulty with Eq. (5) has been removed. The result obtained by starting at $St=1$ hardly differs from it. From this consideration, we may conclude that the nonequilibrium effect in the spectrum (33) plays a very important role at the energy-containing side of the inertial range.

The nonequilibrium effect closely related to the Lagrange derivative D/Dt vanishes in fully developed turbulent flows such as a pipe flow, a channel flow, etc. This is the reason the equilibrium form (5) based on Eq. (25) can give reasonable results in those flows. However, we should note that vanishing of such an effect does not always signify that the turbulence state is in equilibrium. The typical examples are the buffer layer and the central region in a fully developed channel flow. In the latter, the dissipation rate balances with the diffusion rate in the K equation (6), and the momentum is transferred towards the center of a channel through the diffusion effect. The inclusion of such an effect is left for future work.

Let us consider what accounts for the above improved agreement. In the equilibrium turbulent-viscosity model given by Eq. (4) with Eq. (5), R_{ij} is written in terms of single-time quantities. In highly nonstationary turbulence like homogeneous shear turbulence, the effects of the low-wave-number components of motion are transferred to the high-wave-number ones with some time lag. The equilibrium model cannot describe this time-lag effect and estimates the turbulence state on the basis of the information at the latest time. This is the reason the equilibrium model often overestimates the turbulent viscosity in a situation in which the turbulence is intensified rapidly. In the present model (37), the $\partial/\partial t$ term expresses such a time-lag effect in homogeneous shear turbulence.

The $\mathbf{U}\cdot\nabla$ part of D/Dt becomes very important in the situation in which turbulence properties change much in the mainstream direction. Such an example can be seen typically in the case of shock-wave/turbulence interactions. One of their prominent features is that, entirely similarly to homogeneous shear turbulence, the turbulent viscosity should be considerably decreased near a shock wave, compared with the equilibrium one (5). This fact is also supposed to indicate the importance of nonequilibrium effect on the energy spectrum.

Let us refer to the second-order models represented by the model of Launder, Reece, and Rodi [19] in the context of the nonequilibrium turbulent-viscosity model (37). In these models, the modeling of the pressure/velocity-strain correlation function $\langle p'(\partial u_j'/\partial x_i+\partial u_i'/\partial x_j)\rangle$ plays a central role. The second-order model of Launder, Reece, and Rodi type can be derived through the renormalization of a higher-order model of the turbulent-viscosity type [10]. This fact signifies that the latter model can also be obtained by solving the second-order model concerning R_{ij} by an iteration method. From this relationship, we can see that in the second-order model, the D/Dt terms in the turbulent-viscosity-type model are absorbed into the DR_{ij}/Dt term and the modeled expression for $\langle p'(\partial u_j'/\partial x_i+\partial u_i'/\partial x_j\rangle$. In the current second-order modeling, the D/Dt effect has been overlooked in the modeling of $\langle p'(\partial u_j'/\partial x_i+\partial u_i'/\partial x_j\rangle$.

B. Length scale characterizing the turbulent-energy production process

In the present work, we showed that besides l_C and l_D, another length scale l_P defined by Eq. (32) is indispensable for expressing the nonequilibrium property of the inertial-range energy spectrum. Considering that ε is the energy transfer rate from the energy-containing to inertial ranges, we can understand that l_P is related to the dynamical process of the turbulent-energy-production process. Specifically, in the case of homogeneous shear turbulence, a nonequilibrium property of the energy-containing range reflects on $\partial\varepsilon/\partial t$ that expresses the temporal change of the rate of energy transfer to the inertial range.

From Eqs. (17) and (32), we have

$$\frac{l_C}{l_P}=\left(\frac{T_C}{T_P}\right)^{3/2}, \tag{38}$$

where

$$T_C=K/\varepsilon , \tag{39}$$

$$T_P=\varepsilon\Big/\left|\frac{D\varepsilon}{Dt}\right| . \tag{40}$$

Here $T_C(=l_C/K^{1/2})$ is the turnover time of eddies with size l_C, where T_p is a time scale characterizing the dynamical property of the energy-containing range. In the inertial range, the time scale (τ_1) intrinsic to Kolmogorov's scaling law is related to the wave number k as

$$\tau_I=\varepsilon^{-1/3}k^{-2/3} \tag{41a}$$

or

$$k^{-1}=\varepsilon^{1/2}\tau^{3/2} . \tag{41b}$$

Equation (41b) indicates that two time and length scales in the inertial range, which are expressed as $(l_{In},\tau_{In};n=1,2)$, obey

$$\frac{l_{I1}}{l_{I2}}=\left(\frac{\tau_{I1}}{\tau_{I2}}\right)^{3/2} . \tag{42}$$

Equation (42) is originally valid for the inertial range, but it is very similar to Eq. (38). This fact is interesting for the following two reasons. One is that l_C and l_P, which are the length scales characterizing the ranges outside the inertial one, possess the inertial-range length-time relation. Another is that in contrast with Eq. (42), the $-\frac{5}{3}$ power law for the energy spectrum is much affected by the nonequilibrium effect of the turbulent-energy-production process.

Using the DNS results [17,18], we can examine the time development of l_C/l_P, which is given in Fig. 3 and shows that l_P is larger than l_C characterizing the boundary between the energy-containing and inertial ranges. In the production process of turbulent energy, the mean-velocity gradient $\nabla\mathbf{U}$ plays a central role, as can be seen from Eq. (8). Therefore such a process is closely connect-

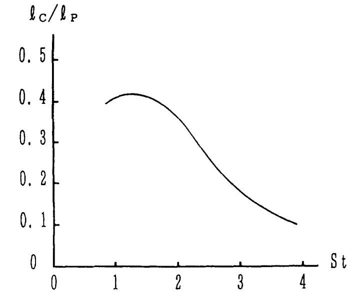

FIG. 3. Time development of l_C/l_P based on the DNS result [17,18].

ed with the low-wave-number properties of turbulence. The relation $l_P > l_C$ is consistent with the fact that l_P characterizes a dynamical property of the turbulent-energy-production process.

Finally, let us refer to the effective energy transfer rate and anisotropy of velocity fluctuation. Equation (33) for $E(k)$ is rewritten as

$$E(k)=K_0\varepsilon_{\rm eff}^{2/3}k^{-5/3}\,, \tag{43}$$

where $\varepsilon_{\rm eff}$ is defined as

$$\varepsilon_{\rm eff}/\varepsilon=\left[1-C_N\,{\rm sgn}\left[\frac{D\varepsilon}{Dt}\right](kl_P)^{-2/3}\right]^{3/2}\,, \tag{44}$$

and may be called the effective energy transfer rate. Let us consider the effect of the sign of $D\varepsilon/Dt$ in Eq. (44). In the case of homogeneous shear turbulence, $D\varepsilon/Dt$ is replaced with $\partial\varepsilon/\partial t$ and is positive. Therefore Eq. (44) signifies that the energy transfer rate decreases effectively, compared with the Kolmogorov counterpart ε. This situation can be explained as follows. In this case, the turbulent energy K continues to be generated in the low-wave-number range, but its transfer towards the inertial range is made with some time lag. As a result, the energy transfer rate decreases effectively, compared with the energy generated and accumulated in the low-wave-number range. The opposite situation occurs in the case of negative $\partial\varepsilon/\partial t$ for isotropic turbulence.

In turbulent shear flows including homogeneous shear turbulence, the velocity fluctuation becomes statistically anisotropic. One of the representative ingredients generating such anisotropy is the mean-velocity gradient $\nabla\mathbf{U}$. In this context, the concept of the energy spectrum is less clear, compared with the case of isotropic turbulence. The nonequilibrium effect appearing in the energy spectrum is distributed among $\langle u_x'^2\rangle$, $\langle u_y'^2\rangle$, $\langle u_z'^2\rangle$, and its importance is unchanged. The anisotropy of turbulent intensities has been discussed in detail in the context of the one-point turbulence modeling [8–10,12].

VII. CONCLUDING REMARKS

In this work, we investigated the nonequilibrium effect of the turbulent-energy-production process on the inertial-range spectrum, with the aid of the TSDIA results. We showed that such properties related to the Lagrange derivative of the energy transfer rate have strong influence on the inertial-range energy spectrum at the energy-containing side. In the context of the one-point turbulence modeling, the resulting deviation of the inertial-range spectrum brings the Lagrange-derivative-related nonequilibrium effect into the turbulent viscosity. This effect resolves the critical difficulty that the equilibrium turbulent-viscosity model encounters in highly time-dependent turbulent flows such as homogeneous shear turbulence. In the context of the relationship with the intermittency effect on the energy spectrum that arises from fluctuation of the local energy-dissipation rate, its importance increases at the dissipation-range side of the inertial range, contrary to the present nonequilibrium effect.

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